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ON THE SOLUTION OF OPTIMIZATION PROBLEMS WITH SINGULARITIES†

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Abstract—We derive a complete set of necessary optimality conditions for a class of variational problems whose extremal solutions are associated with singularities. The use of these conditions is illustrated by two examples involving the optimization of the shapes of elastic bodies with stiffness and stability constraints.

INTRODUCTION

A typical feature of the optimal design of a number of structural elements is the appearance of singular points (lines in the case of two dimensional structures). At these points (lines) the coefficients of the higher order derivatives in the differential equations governing the optimization problem vanish. It is well-known, that the behaviour of the optimal design (shape of the element) in the neighbourhood of these singular points (lines) has a significant influence on the integral characteristics of the optimal solution. To illustrate this, we point to the fundamental results pertaining to the investigation of solid elastic plates with a sharp edge (see, e.g. [1]). Thus, if the thickness of the plate (in the vicinity of the sharp edge) tends to vanish at a certain rate, the positive definiteness of the operator governing the flexure of the plate is violated, while the spectrum of the natural frequencies of the plate ceases to be discrete. A study of the asymptotic behaviour of the optimal solutions is also interesting from the point of developing efficient numerical techniques for the solution of optimal design problems.

In connexion with the problem of maximizing the fundamental frequency of vibrating beams Niordson[2] made a beginning in studying the singular behaviour of the optimal solution. Niordson[2], as well as the works of Karihaloo and Niordson[3, 4] and Olhoff[5] considered cases where the position of the singular points coincides with the ends of the interval in which the solution is sought. The behaviour of the optimal solutions in the neighbourhood of inner singular points was studied by Olhoff[6, 7] and Karihaloo and Niordson[8].

In the case of inner singular points the number of the unknowns to be determined from the solution of the optimization problem is increased by an amount equal to the coordinates of these points. Significantly, the coordinates of the singular points are not defined by the equilibrium equations or the Euler equations in the governing variables. In the earlier investigations the existence of the so-called "free" parameters (coordinates of the singular points) was either construed as a multiple extremality of the optimization problem or these parameters were eliminated by making some additional assumptions. Thus, Prager and Taylor[9], in studying the problem of maximizing the bending stiffness of a beam, removed this "non-uniqueness" by demanding continuity of the first derivative of the deflection function at the singular points. In [10], the same continuity condition was obtained not by imposing a certain additional demand or assumption, but as a result of maximizing the functional of the optimization problem with respect to the "free" parameter.

As will be shown later, this "non-uniqueness" is a consequence of the fact that an incomplete set of necessary stationary conditions was employed in these investigations. In order to allow for the possibility of the appearance of singular points in the optimal solution it is

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important to include in the investigation the Weierstrass–Erdmann corner conditions. The latter allow us to uniquely determine the position of the singular points. It transpires that the requirement of continuity of the first derivative in the optimal design problems with stiffness and stability constraints, assumed intuitively in [9], follows from an application of the Weierstrass–Erdmann conditions to the points where there is a jump in the first derivative of the deflection function. This very condition is valid for some other classes of problems, too.

1. NECESSARY STATIONARY CONDITIONS IN VARIATIONAL PROBLEMS WITH NON-ADDITIVE FUNCTIONALS AND NON-SMOOTH EXTREMALS

Let us consider the following variational problem. It is required to find the vector-function $u(x) = [u^1(x), \ldots, u^n(x)]$ in the interval $[x_0, x_1]$ that renders the following non-additive functional stationary

$$J = F(J_1, J_2, \ldots, J_s), \tag{1}$$

where

$$J_k = \int_{x_0}^{x_1} f^k(x, u, u_x, u_{xx})\,\mathrm{d}x, \qquad k = 1, 2, \ldots, s, \tag{2}$$

u_x denotes $\mathrm{d}u/\mathrm{d}x$, and so on.

For the sake of definiteness we assume that the unknown vector-function is subject, at the ends of the interval $[x_0, x_1]$, to $4n$ known boundary conditions, $u^i(x_0) = \alpha_0^i$, $u_x^i(x_0) = \beta_0^i$, $u^i(x_1) = \alpha_1^i$, $u_x^i(x_1) = \beta_1^i (i = 1, 2, \ldots, n)$; where α_0^i, β_0^i, α_1^i, β_1^i, n and s are given constants, and F and f^k are given functions of their arguments.

The extremum of problem (1), (2) is sought among a class of continuous functions with piece-wise continuous first derivative. We assume that there is a jump in the derivative at some point $x = x_*$ $(x_0 < x_* < x_1)$ whose position is not known beforehand but is to be determined from the stationary condition of the functional J.

Before passing to the derivation of the necessary stationary conditions using classical variational techniques we observe that these conditions can also be obtained from the fundamental equations of optimal control theory (Appendix).

For the derivation of the necessary stationary conditions let us express each of the integrals J_k $(k = 1, 2, \ldots, s)$ as a sum of two integrals, $J_k = J_k^1 + J_k^2$, taken over the intervals $[x_0, x_*]$ and $[x_*, x_1]$, respectively, and write an expression for the first variation of each of these integrals. Here it is assumed that there is only one singular point in the interval $[x_0, x_1]$. The expressions for δJ_k are obtained by summing δJ_k^1 and δJ_k^2 having regard to the continuity of u at the point $x = x_*$. Furthermore, let us expand the function F in powers of δJ_k and retain only terms of the first order of magnitude. After some transformations, we have

$$\begin{aligned}\delta J = \sum_{i=1}^{n}\Bigg\{&\int_{x_0}^{x_*}\sum_{k=1}^{s}\frac{\partial F}{\partial J_k}\left(\frac{\partial f^k}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f^k}{\partial u_x^i} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial f^k}{\partial u_{xx}^i}\right)\delta u^i\,\mathrm{d}x + \\ &+\int_{x_*}^{x_1}\sum_{k=1}^{s}\frac{\partial F}{\partial J_k}\left(\frac{\partial f^k}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f^k}{\partial u_x^i} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial f^k}{\partial u_{xx}^i}\right)\delta u^i\,\mathrm{d}x - \\ &-\delta u^i\sum_{k=1}^{s}\frac{\partial F}{\partial J_k}\left[\frac{\partial f^k}{\partial u_x^i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f^k}{\partial u_{xx}^i}\right]_-^+ - \left[\delta u_x^i\sum_{k=1}^{s}\frac{\partial F}{\partial J_k}\frac{\partial f^k}{\partial u_{xx}^i}\right]_-^+\Bigg\} \\ &-\delta x_*\sum_{k=1}^{s}\frac{\partial F}{\partial J_k}\left[f^k - \sum_{i=1}^{n}\left(u_x^i\frac{\partial f^k}{\partial u_x^i} + u_{xx}^i\frac{\partial f^k}{\partial u_{xx}^i} - u_x^i\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f^k}{\partial u_{xx}^i}\right)\right]_-^+.\end{aligned}$$

Here, and in the sequel, "+" and "−" refer to the values of the corresponding quantities evaluated at $x = x_* + 0$ and $x = x_* - 0$, respectively, while $[\cdots]_-^+$ denotes the difference in the limiting values of the quantity enclosed in the square brackets, i.e. $[\cdots]_-^+ = (\cdots)^+ - (\cdots)^-$.

Equating δJ to zero and keeping in mind the arbitrariness of functions $u^i(x)$ in the interval $[x_0, x_1]$ and of δx_*, $(\delta u_x^i)^+$, $(\delta u_x^i)^-$, $\delta u^i = (\delta u^i)^+ = (\delta u^i)^-$, we obtain the Euler equation

$$\sum_{k=1}^{s}\frac{\partial F}{\partial J_k}\left(\frac{\partial f^k}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f^k}{\partial u_x^i} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial f^k}{\partial u_{xx}^i}\right) = 0, \qquad i = 1, 2, \ldots, n \tag{3}$$

and the Weierstrass–Erdmann conditions at the point $x = x_*$ where there is a discontinuity in the derivatives

$$\left(\sum_{k=1}^{s} \frac{\partial F}{\partial J_k} \frac{\partial f^k}{\partial u^i_{xx}}\right)^+ = \left(\sum_{k=1}^{s} \frac{\partial F}{\partial J_k} \frac{\partial f^k}{\partial u^i_{xx}}\right)^- = 0, \quad i = 1, 2, \ldots, n \tag{4}$$

$$\sum_{k=1}^{s} \frac{\partial F}{\partial J_k} \left[\frac{\partial f^k}{\partial u_x{}^i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f^k}{\partial u^i_{xx}}\right]_-^+ = 0, \tag{5}$$

$$\sum_{k=1}^{s} \frac{\partial F}{\partial J_k} \left[f^k - \sum_{i=1}^{n} \left\{u^i_{xx} \frac{\partial f^k}{\partial u^i_{xx}} + u_x{}^i \left(\frac{\partial f^k}{\partial u_x{}^i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f^k}{\partial u^i_{xx}}\right)\right\}\right]_-^+ = 0. \tag{6}$$

The derivative $\partial F/\partial J_k = F_{J_k}(J_1, J_2, \ldots, J_s)$ in the Euler eqn (3) and the conditions (4)–(6) are evaluated at the values of the integrals $J_1, J_2, \ldots, J_s$ that correspond to the extremum of the variational problems (1), (2). Consequently, (3) may be regarded as integro-differential equations.† The conditions (4)–(6) are easily generalized to the case when the jumps in the derivatives of various components of vector u occur at different values of the independent variable.

The Euler eqns (3) written in the intervals $[x_0, x_*)$, $(x_*, x_1]$ form a system of $2n$ integro-differential equations each of which is of the fourth order. For determining the $8n$ constants of integration of this system and the unknown quantity x_* we have $4n$ boundary conditions at the ends of the interval $[x_0, x_1]$, $(3n + 1)$ Weierstrass-Erdmann conditions (4)–(6) and n continuity conditions of the components of vector-function u at the point $x = x_*$, i.e. in all $(8n + 1)$ conditions for determining an equal number of unknowns. We thus arrive at a closed (multipoint) boundary value problem for a system of integro-differential equations.

It should be noted that the above considerations were essentially independent of the type of boundary conditions. Therefore, the relations (3)–(6) will be used later for boundary value problems of a different type, too.

In the examples considered below the minimum of the functional F is sought subject to certain isoperimetric conditions. In deriving Euler equations and other necessary stationary conditions these latter are taken into account in the usual manner through the use of Lagrange multipliers.

2. OPTIMAL BENDING RIGIDITY OF A PLATE

Let us consider the problem of an infinitely long (along y-direction) rectangular plate built-in along the edges $|x| = l$ and subject to a uniformly distributed load (intensity P) along the line of symmetry $x = 0$. It is evident that the deflection u and other mechanical parameters are independent of the co-ordinate y. Consequently, we restrict ourselves to the plane $y = 0$. The load intensity P_* necessary to cause the point $x = 0$, $y = 0$ to deflect by a given amount u_0 will be treated as a measure of the bending stiffness of the plate. Let $h(x)$ denote the thickness variation of the plate. The problem under consideration consists in finding the continuous thickness variation function $h(x)$ that satisfies the isoperimetric equality

$$\int_{-l}^{l} h(x)\,\mathrm{d}x = S \tag{7}$$

and maximizes the integral

$$P_* = \max_h P(h) = \max_h \min_u \frac{K_m}{u_0} \int_{-l}^{l} h^m u_{xx}^2\,\mathrm{d}x \tag{8}$$

†The integro-differential form of writing the necessary stationary conditions is convenient for developing efficient numerical schemes for solving variational problems. The simplest iterative algorithm may be obtained, if the quantities F_{J_k} entering the expressions (3)–(6) are determined from the m-th approximation

$$F_{J_k} = F_{J_k}(J_1(u^{(m)}), \ldots, J_s(u^{(m)}))$$

to the unknown vector-function u, whilst in determining the rest of the quantities f^k, $\partial f^k/\partial u^i$, $\partial f^k/\partial u_x{}^i$, $\partial f^k/\partial u^i_{xx}, \ldots$, u is set to equal to $u^{(m+1)}$.

subject to the conditions

$$u(0)=u_0,\quad u(-l)=u(l)=u_x(l)=u_x(-l)=0, \tag{9}$$

where u_0 and S are given numbers. The values of m and K_m depend upon the type of plate. The particular cases of $m=1$, $K_1=EH^2/4(1-\nu^2)$ and $m=3$, $E_3=E/12(1-\nu^2)$ considered below correspond to sandwich and solid plates, respectively. Here, E is the modulus of elasticity, ν—Poisson's ratio and H—the constant core thickness of the sandwich plate. For a solid plate $(m=3)h(x)$ refers to the thickness variation, while for a sandwich plate $\frac{1}{2}h(x)$ refers to the variable thickness of facets. (For a detailed discussion of the problem (7)–(9), see [11]).

The necessary stationary conditions for the problem (7)–(9) with respect to u and h are easily shown to be

$$(h^m u_{xx})_{xx}=0,\quad h^{m-1}u_{xx}^2=\lambda^2, \tag{10}$$

where λ^2 is a number. The first of these conditions should be satisfied throughout the interval $[-l,l]$ except for $x=0$ and the points x_{*i} of the jump in the derivatives, whereas the second condition should be satisfied throughout this interval. At the points x_{*i} where there is a jump in the derivatives of the function $u(x)$ the optimal solution should fulfill Weierstrass–Erdmann conditions (4)–(6), which in the present problem take the form

$$(h^m u_{xx})^+=(h^m u_{xx})^-=0,\quad [(h^m u_{xx})_x]_-^+=0, \tag{11}$$

$$[-h^m u_{xx}^2+2u_x(h^m u_{xx})_x]_-^+=0. \tag{12}$$

It should be noted that, if the singular point coincides with $x=0$ (point of application of the load) at which the condition $u=u_0$ is specified, then the second of the two conditions (11) and the condition (12) drop out (for in the expression for the variation of the functional the corresponding terms are absent). Therefore, the case when $x_{*i}=0$ needs separate treatment.

The conditions (11) have a simple physical meaning. At the singular points the bending moment $M=h^m u_{xx}$ vanishes, but the shear force $Q=(h^m u_{xx})_x$ varies continuously.

It follows from (10) and (11) that the thickness function $h(x)$ satisfies the following conditions at the singular points

$$h^+=h^-=0,\quad (h^{(m+1)/2})_x^-=-(h^{(m+1)/2})_x^+. \tag{13}$$

Let us analyse the relations (13). The first term in (12), in view of the optimality conditions (10), takes the form $h^m u_{xx}^2=\lambda^2 h$ and is continuous, since the function $h(x)$ is continuous. Thus (12) reduces to

$$[u_x(h^m u_{xx})_x]_-^+=0. \tag{14}$$

Keeping in view the second of the two relations (11), we can conclude that the condition (14) will be observed, if either

$$(u_x)^+=(u_x)^-, \tag{15}$$

or

$$(h^m u_{xx})_x^+=(h^m u_{xx})_x^-=0. \tag{16}$$

That (14) are the necessary optimality conditions was also shown by Masur[12] and their sufficiency is implied in [9] for the sandwich plate.

Let us show that (16) cannot be realised in the optimal solution sought here. In fact, from the relations (10), (13) and (16) we have the following differential equation and boundary conditions for determining h

$$\left.\begin{aligned}(h^{(m+1)/2})_{xx}=0,\quad (x\neq x_{*i})\\ h^+=h^-=(h^{(m+1)/2})_x^+=(h^{(m+1)/2})_x^-=0.\end{aligned}\right\} \tag{17}$$

It is easily seen that the above boundary-value problem for the function $h(x)$ does not have a non-trivial solution. Consequently, at the singular points the condition (15) must be satisfied. The optimal thickness distribution should be such that the deflection function has a continuous first derivative, while the higher order derivatives may be discontinuous at the singular points ($x_{*i} \neq 0$).

In the present optimization problem elementary considerations lead to the conclusion that the number of singular points cannot exceed two. In order to prove this let us assume the contrary, viz. that the optimal solution has three or more singular points. From the symmetry of the problem it is evident that the singular points in the interval $[-l, l]$ should be located symmetrically relative to the point of application of the load $x = 0$. By making a judicious choice of comparison functions $u(x)$ in the variational problem (7)–(9) it is easy to show that, for any thickness distribution $h(x)$, the functional being maximized is equal to zero $P_* = P(h) = 0$ (as an admissible function $u(x)$ we may, for example, choose a continuous, piecewise linear function for which $\int_{-l}^{l} h^m u_{xx}^2 \, dx = 0$). This fact has a simple physical meaning. If the points with $h = 0$ are treated as hinges, we have a system which cannot be in equilibrium whatever the value of P. Therefore, the number of singular points cannot exceed two.

2.1 *Sandwich plate* $m = 1$

In this case the equilibrium equations and the optimality condition (10) take the form

$$h_{xx} = 0, \quad u_{xx}^2 = \lambda^2. \tag{18}$$

From the second of the two eqns (18) and the conditions (9) it is easily seen that a continuous, twice differentiable function $u(x)$ does not exist. Therefore, we shall seek a solution $u(x)$ in the class of functions having discontinuous derivatives. From symmetry considerations it follows that the singular points in the optimal solution are located symmetrically relative to the origin ($x_{*1} = -x_{*2}$). All the formulae are thus presented only for the region $[-l, 0]$. From the second eqn (18) and the conditions (9) the deflection function may be obtained to within a constant x_{*1}. However, by utilizing the conditions (15), we obtain $x_{*1} = -l/2$. The expressions for the deflection function and the load intensity (functional being maximized) are

$$\begin{aligned} u &= 2u_0\left(1 + \frac{x}{l}\right)^2, \quad (-l \leq x \leq -l/2) \\ u &= u_0\left(1 - \frac{2x^2}{l^2}\right), \quad (-l/2 \leq x \leq 0) \\ \lambda &= 4u_0 l^{-2}, \\ P_* &= K_1 S \lambda^2 u_0^{-1} = 16 K_1 S u_0 l^{-4}. \end{aligned} \tag{19}$$

It may be noted that P_* can be found without a knowledge of $h(x)$. The latter is found from the first eqn (18), the isoperimetric condition (7) and the relations at the singular points (13)

$$\begin{aligned} h &= -\frac{S}{l^2}\left(x + \frac{l}{2}\right), \quad (-l \leq x \leq -l/2) \\ h &= \frac{S}{l^2}\left(x + \frac{l}{2}\right), \quad (-l/2 \leq x \leq 0). \end{aligned} \tag{20}$$

Having considered the special case $x_{*1} = -x_{*2} = 0$ separately and performed corresponding calculations, we obtain a lesser value of the load $P_* = 4K_1 S u_0 l^{-4}$. Consequently, the solution (19), (20) is the optimal.† The result is well-known[9].

†It should be mentioned that in the given case ($m = 1$) the stationary conditions employed to arrive at the solution (19), (20) are not only necessary, but also sufficient conditions of optimality[9].

2.2 *Solid plate* $m = 3$

As in the preceding example, we restrict ourselves to the interval $-l \leq x \leq 0$. From the relations (10) we obtain the following equation describing the deflection distribution in the optimal plate

$$(u_{xx}^{-2})_{xx} = 0, \qquad (-l < x < x_{*1}, x_{*1} < x < 0). \tag{21}$$

Let us integrate eqn (21) in each of the intervals $[-l, x_{*1}]$ and $[x_{*1}, 0]$ and determine the eight constants of integration from the boundary conditions and the relations at the singular points

$$u(-l) = u_x(-l) = u_x(0) = 0, \quad u(0) = u_0,$$
$$(u_{xx}^{-2})^- = (u_{xx}^{-2})^+ = 0, \quad (u_{xx}^{-2})_x^- = -(u_{xx}^{-2})_x^+, \quad [u]_-^+ = 0. \tag{22}$$

The last four relations follow from (10), (11) and the continuity condition of the function $u(x)$. The deflection function will depend on the co-ordinate of the singular point x_{*1} which is determined from the condition (15) and turns out to be $x_{*1} = -l/2$. The deflection function finally takes the form

$$u = \frac{3\sqrt{(2)}u_0}{l^{3/2}}\left[\sqrt{\left(\frac{l}{2}\right)}(x+l) + \frac{2}{3}\left\{\left(-\frac{l}{2} - x\right)^{3/2} - \left(\frac{l}{2}\right)^{3/2}\right\}\right], \quad -l \leq x \leq -l/2$$
$$u = \frac{3\sqrt{(2)}u_0}{l^{3/2}}\left[\sqrt{\left(\frac{l}{2}\right)}x + \frac{2}{3}\left\{\left(\frac{l}{2}\right)^{3/2} - (x + l/2)^{3/2}\right\}\right] + u_0, \quad -l/2 \leq x \leq 0. \tag{23}$$

The optimal thickness distribution and the corresponding value of load intensity (a well-known result[9]), found from the relations (7), (8), (10) and (23), are

$$h = \frac{3S}{4l}\sqrt{\left(-1 - \frac{2x}{l}\right)}, \qquad -l \leq x \leq -l/2$$
$$h = \frac{3S}{4l}\sqrt{\left[\left(1 + \frac{2x}{l}\right)\right]}, \quad -l/2 \leq x \leq 0 \tag{24}$$
$$P_* = \frac{81K_3u_0S^3}{4l^6}.$$

Having made similar calculations for the special case $x_{*1} = x_{*2} = 0$, we arrive at a lesser value of the load intensity

$$P_* = 81K_3u_0S^3/64l^6.\dagger$$

3. OPTIMAL SHAPE OF AN AXIALLY COMPRESSED BAR

As an additional illustration let us consider the problem of stability of an axially compressed (load P) elastic bar of length l. The bar is assumed to be built-in at $x = 0$ and pinned at $x = l$. Let $h(x)$ denote the thickness variation along the length of the bar and $u(x)$ the deflection from the initial straight configration. The problem consists in finding the shape of the bar that can sustain the maximum load without loss of stability. By using Rayleigh principle the problem is

†A study of the variation in P for any arbitrary position of the singular point $x_{*1}(-l \leq x_{*1} \leq 0)$, i.e. in the absence of the optimality condition (15) gives

$$P = \frac{81K_3S^3u_0}{64}[(x_{*1} + l)^{3/2} + (-x_{*1})^{3/2}]^{-4},$$

whence it directly follows that the maximum of $P(x_{*1})$ is achieved at $x_{*1} = -l/2$, and is given by $P = 81K_3S^3u_0/4l^6$. It may be noted that for $x_{*1} \to 0$ as well as $x_{*1} \to -l$ the value of P decreases monotonically and tends to approach in both cases to $P = 81K_3S^3u_0/64l^6$.

reduced to finding the continuous thickness distribution that satisfies the isoperimetric condition

$$\int_0^l h(x)\,\mathrm{d}x = S \tag{25}$$

and maximizes the functional (critical load)

$$P_* = \max_h P(h) = \max_h \min_u \frac{K_m \int_0^l h^m u_{xx}^2\,\mathrm{d}x}{\int_0^l u_x^{\,2}\,\mathrm{d}x} \tag{26}$$

subject to the conditions

$$u(0) = u_x(0) = u(l) = (h^m u_{xx})|_{x=l} = 0. \tag{27}$$

In the given problem the complete set of necessary optimality conditions (3)–(6) takes the following form

$$h^{m-1}u_{xx}^2 = \lambda^2, \quad K_m(h^m u_{xx})_{xx} + Pu_{xx} = 0, \tag{28}$$

$$(h^m u_{xx})^+ = (h^m u_{xx})^- = 0, \quad [K_m(h^m u_{xx})_x + Pu_x]_-^{\,+} = 0, \tag{29}$$

$$[Pu_x^{\,2} - K_m h^{m-1}u_{xx}^2 + 2u_x(K_m h^m u_{xx})_x]_-^+ = 0. \tag{30}$$

(The parameter K_m here is obviously different from the previous one.) Conditions (29) and (30) must be fulfilled at the points of discontinuity $x = x_{*i}$ in the derivatives. Following arguments similar to those used previously it is easy to show that the optimal solution does not have more than one singular point and that conditions (30), together with (28) and (29), lead to the equality (15) which will be used later to determine the position of the singular point.

For simplicity we present the solution for the special case $m = 1$. From the optimality condition $u_{xx}^2 = \lambda^2$ and the boundary conditions (27) it is easily verified that there does not exist a non-trivial, ($u \not\equiv 0$), twice differentiable function. Thus, the solution will have discontinuities in the derivatives, and in seeking it the conditions (15), (29) must be used together with the eqns (28). At the point of discontinuity $x = x_*$ in the derivatives of $u(x)$, as is clear from the relations (29) and the first eqn (28), the following equalities should be satisfied

$$h^+ = h^- = 0, \quad (h_x)^+ = -(h_x)^-. \tag{31}$$

For determining the deflection function of the optimal bar and the position of the singular point we utilize the equation $u_{xx}^2 = \lambda^2$ ($u_{xx} = -\lambda$ for $x \leq x_*$ and $u_{xx} = \lambda$ for $x_* \leq x \leq l$), the first three boundary conditions (27) and the compatibility conditions at the singular point, viz. $u^+ = u^-$, $(u_x)^+ = (u_x)^-$. Performing some elementary calculations, we find the deflection function

$$u = -\frac{\lambda x^2}{2}, \quad 0 \leq x \leq x_* \tag{32}$$

$$u = \frac{\lambda x}{2}[x - 2(2-\sqrt{2})l] + \frac{\lambda l^2}{2}(2 - 2\sqrt{2}), \quad x_* \leq x \leq l,$$

and the position of the singular point $x_* = l(\sqrt{(2)} - 1)/\sqrt{(2)}$. The critical load calculated from (26), (28) and (32) is given by $P = 6K_mS/(3\sqrt{(2)} - 4)l^3$. The thickness function $h(x)$ in each of the intervals $0 < x < x_*$, $x_* < x < l$ is determined from (28) with $m = 1$, viz. $h_{xx} + PK_1^{-1} = 0$. For evaluating the constants of integration we have the three conditions (31) at the singular point. The fourth condition $h(l) = 0$ follows from the equations $u_{xx}^2 = \lambda^2$ and $(hu_{xx})|_{x=l} = 0$. Finally, we get

$$h = \frac{3S}{(3\sqrt{(2)} - 4)l^3}\left[l\left(1 - \frac{1}{\sqrt{2}}\right) - x\right][x + l(\sqrt{2} - 1)], \quad 0 < x < x_*$$

$$h = \frac{3S}{(3\sqrt{2} - 4)l^3}\left[x - l\left(1 - \frac{1}{\sqrt{2}}\right)\right](l - x), \quad x_* < x < l. \tag{33}$$

4. CONCLUSIONS

From the general mathematical considerations and the solution of illustrative examples it is evident that to pick out the unique optimal solution there is no need for making any additional assumptions, the position of singular points being determined by Weierstrass–Erdmann corner conditions.

In the present work we considered the simplest structural optimization problems which were described by ordinary differential equations and lent themselves to an analytical treatment. In more complex situations when an analytical solution is not feasible and one has to resort to numerical techniques, the use of Weierstrass–Erdmann conditions becomes that much more relevant.

In the examples treated above the number of singular points does not exceed two. However, already in the optimal design problem of a bar on an elastic foundation with stability constraint the instability mode can have any number of such points depending upon the stiffness of the foundation. In this problem, too, Weierstrass–Erdmann conditions permit us to determine the number and position of the singular points.

Finally, in the optimal design of two-dimensional structures (plates and shells) where the equations degenerate along whole lines the solution may be sought by using, for example, gradient methods which utilize Weierstrass–Erdmann conditions.

REFERENCES

1. S. G. Mikhlin, *Variational Methods in Mathematical Physics* (In Russian), p. 265. Nauka, Moscow (The English translation of an earlier edition of this book does not contain the problem referred to in the text).
2. F. I. Niordson, *Quart. Appl. Maths* **23**(1), (1965).
3. B. L. Karihaloo and F. I. Niordson, *J. Optimization Theory & Applications* **11**(6), (1973).
4. B. L. Karihaloo and F. I. Niordson, *Archives of Mech.*, Vol. 24, No. 5–6, Warsaw, (1972).
5. N. Olhoff, *Int. J. Solids Structures* **6**(2), (1970).
6. N. Olhoff, *Proc. IUTAM Symp. Optimization in Structural Design*, Warsaw 1973, (Edited by A. Sawczuk and Z. Mroz). Springer Verlag, Berlin (1975).
7. N. Olhoff, *J. Struct. Mech.* **4**(1), (1976).
8. B. L. Karihaloo and F. I. Niordson, *Proc. IUTAM Symp. Optimization in Structural Design* (Warsaw 1973. Springer Verlag, Berlin (1975).
9. W. Prager and J. E. Taylor, *J. Appl. Mech.* **35**(3), (1968).
10. E. F. Masur, Written discussion on Ref. [6], *Proc. IUTAM Symp. Optimization in Structural Design*. Springer Verlag, Berlin (1975).
11. N. V. Banichuk, *Izv. AN USSR*, "*Mekhanika Tverdogo Tela*" No. 5 (1975). (English Translation: *J. Mechanics of Solids*).
12. E. F. Masur, *Proc. IUTAM Symp. Optimization in Structural Design*, Warsaw 1973, (Edited by A. Sawczuk and Z. Mroz). Springer Verlag, Berlin (1975).
13. A. E. Bryson and Y.-C. Ho, *Applied Optimal Control*. Blaisdell, Waltham, Mass. (1968).

APPENDIX

In the main body of the text the stationary conditions of a functional were obtained by using classical variational calculus. We present here an alternate derivation of these conditions by employing concepts from the optimal control theory (see, e.g. [13]). However, for the sake of brevity we consider only the case of a scalar function $u(x)$.

Let us introduce a new set of variables $z^1, z^2, \dots z^s, z^{s+1}, z^{s+2}$ governed by the following system of differential equations and boundary conditions

$$\begin{aligned} z_x^i &= f^i(x, z^{s+1}, z^{s+2}), z^i(x_0) = 0, \quad i = 1, 2, \dots, s \\ z_x^{s+1} &= z^{s+2}, z^{s+1}(x_0) = \alpha_0, z^{s+1}(x_1) = \alpha_1 \\ z_x^{s+2} &= v, z^{s+2}(x_0) = \beta_0, z^{s+2}(x_1) = \beta_1. \end{aligned} \tag{A1}$$

Here $z^{s+2} = u_x$, and the variable v which is related to u through $v = z_x^{s+2} = u_{xx}$ plays the role of the control function. In terms of the new variables the functional which is being maximized takes the form

$$J = F(z^1(x_1), z^2(x_1), \dots, z^s(x_1)). \tag{A2}$$

Thus, we arrive at the optimal control problem for the system (A1) whose functional is a function of the phase co-ordinates z^i ($i = 1, 2, \dots, s$) at a finite point $x = x_1$ (the Maier problem).

The Hamiltonian for the system (A1) and the functional (A2) is defined by

$$H = \sum_{i=1}^{s} p^i f^i + p^{s+1} z^{s+2} + p^{s+2} v, \tag{A3}$$

where p^i denote conjugate variables which satisfy the following system of differential equations

$$p_x^i = 0, \quad i = 1, 2, \ldots, s$$
$$p_x^{s+1} = -\sum_{i=1}^{s} p^i \frac{\partial f^i}{\partial z^{s+1}}, \quad p_x^{s+2} = -\left(\sum_{i=1}^{s} p^i \frac{\partial f^i}{\partial z^{s+2}} + p^{s+1}\right). \tag{A4}$$

From (A4) it follows that the conjugate functions p^i are constants. At the point $x = x_1$ these functions satisfy transversality conditions $p^i(x_1) = (\partial F/\partial z^i)|_{x=x_1}$. Consequently, the following relations hold at the extremals

$$p^i = \left(\frac{\partial F}{\partial z^i}\right)\bigg|_{x=x_1}, \quad i = 1, 2, \ldots, s. \tag{A5}$$

For the problem under consideration the optimality condition $\partial H/\partial v = 0$ takes the form

$$\sum_{i=1}^{s} p^i \frac{\partial f^i}{\partial v} + p^{s+2} = 0. \tag{A6}$$

Differentiating (A6) with respect to x and eliminating from the resulting expression p_{xx}^{s+2} by using the last eqn (A4), we get

$$\sum_{i=1}^{s} p^i \left[\left(\frac{\partial f^i}{\partial v}\right)_{xx} - \left(\frac{\partial f^i}{\partial z^{s+2}}\right)_x + \frac{\partial f^i}{\partial z^{s+1}}\right] = 0.$$

Furthermore, by taking into account (A5) and returning to the original variables, we get the equation

$$\sum_{i=1}^{s} F_{J_i}\left[\left(\frac{\partial f^i}{\partial u_{xx}}\right)_{xx} - \left(\frac{\partial f^i}{\partial u_x}\right)_x + \frac{\partial f^i}{\partial u}\right] = 0, \tag{A7}$$

which coincides with the Euler eqn (3) derived in the text.

If at some point $x = x_*$ of the optimal solution there is a discontinuity in the variable z^{s+2}, i.e. in the first derivative of the function $u(x)$, then at this point the following necessary optimality conditions must be fulfilled

$$\begin{aligned}
&[p^i(x_*)]^- = [p^i(x_*)]^+, \quad i = 1, 2, \ldots, s+1 \\
&[p^{s+2}(x_*)]^- = [p^{s+2}(x_*)]^+ = 0, \\
&[H(x_*)]^- = [H(x_*)]^+, \quad [H_v(x_*)]^- = [H_v(x_*)]^+.
\end{aligned} \tag{A8}$$

By using the relations (A3)–(A6), the conditions (A8) may be written in the following form

$$\left(\sum_{i=1}^{s} \frac{\partial F}{\partial z^i}\frac{\partial f^i}{\partial v}\right)^- = \left(\sum_{i=1}^{s} \frac{\partial F}{\partial z^i}\frac{\partial f^i}{\partial v}\right)^+ = 0, \tag{A9}$$

$$\left[\sum_{i=1}^{s} \frac{\partial F}{\partial z^i}\left(\left(\frac{\partial f^i}{\partial v}\right)_x - \frac{\partial f^i}{\partial z^{s+2}}\right)\right]_-^+ = 0, \tag{A10}$$

$$\left[\sum_{i=1}^{s} \frac{\partial F}{\partial z^i}\left\{f^i - v\frac{\partial f^i}{\partial v} - z^{s+2}\left(\frac{\partial f^i}{\partial z^{s+2}} - \left(\frac{\partial f^i}{\partial v}\right)_x\right)\right\}\right]_-^+ = 0. \tag{A11}$$

It is easily seen that in the original variables conditions (A9)–(A11) coincide with (4)–(6) derived in the text.